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On results of G. Brosch and T.C. Alzahary

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Abstract

We prove a uniqueness theorem for two non-constant meromorphic functions sharing three values which improves a recent result of T.C. Alzahary. As a consequence of our main result we also improve a theorem of G. Brosch.
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1. Introduction, definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if the a-points of f and g coincide in locations and multiplicities. If we do not consider the multiplicities, we say that f and g share the value a IM (ignoring multiplicities).

Nevanlinna's four value uniqueness theorem states that if two distinct non-constant meromorphic functions share four values CM, then one is a bilinear transformation of the other (cf. [16], [17, p. 218]). In 1983 G.G. Gundersen proved that in Nevanlinna's four value theorem it is possible to relax the nature of sharing of two values from CM to IM (cf. [5]). In 1989 G. Brosch (cf. [3], [17, p. 329]) improved the four value theorem in another direction and proved the following result.

Theorem A. *(See [3].) Let f and g be two distinct non-constant meromorphic functions sharing $0, 1, \infty$ CM. Let a, b be two complex numbers such that $a, b \notin \{0, 1, \infty\}$. If $f - a$ and $g - b$ share 0 IM, then f is a bilinear transformation of g.*

Considering the following example of G.G. Gundersen [4] one can verify that it is not possible in Theorem A to replace all the three CM shared values by IM shared values.

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Example 1.1. (See [4].) Let $f=\frac{e^z+1}{(e^z-1)^2}$ and $g=\frac{(e^z+1)^2}{8(e^z-1)}$. Then f and g share 0, 1, ∞ IM. Also we see that $f+\frac{1}{2}$ and $g-\frac{1}{4}$ share 0 CM but f is not a bilinear transformation of g.

Using the notion of weighted value sharing recently some attempts have been made to relax the nature of sharing the values in Theorem A (cf. [2,14]). We now explain the idea of weighted value sharing which measures how close a shared value is to being shared IM or to being shared CM.

Definition 1.1. (See [7,8].) Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\leqslant k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f-a$ with multiplicity m $(\leqslant k)$ if and only if it is a zero of $g-a$ with multiplicity m $(\leqslant k)$ and z_0 is a zero of $f-a$ with multiplicity m $(>k)$ if and only if it is a zero of $g-a$ with multiplicity n $(>k)$ where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for all integers p, $0\leqslant p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞), respectively.

Recently T.C. Alzahary [1] worked on Brosch's result and investigated the possibility of relaxing the nature of value sharing by f and g and that of sharing the value 0 by $f-a$ and $g-b$. We require the following definitions to state Alzahary's theorem.

Definition 1.2. Let k be a positive integer or infinity. We denote by $\overline{E}_{k)}(a;f)$ the set of all distinct a-points of f with multiplicities not exceeding k. Also by $\overline{E}(a;f)$ we denote $\overline{E}_{\infty)}(a;f)$.

Definition 1.3. Let f be a meromorphic function and $a\in\mathbb{C}\cup\{\infty\}$. For a positive integer p we denote by $N(r,a;f\mid\leqslant p)$ $(\overline{N}(r,a;f\mid\leqslant p))$ the counting function (reduced counting function) of those a-points of f whose multiplicities are less than or equal to p.

We now state the result of Alzahary [1].

Theorem B. *(See [1].) Let f and g be two non-constant meromorphic functions sharing $(a_1,1)$, (a_2,∞) and (a_3,∞), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$. Further let a and b be two complex numbers such that $a,b\notin\{0,1,\infty\}$ and $\overline{E}_{2)}(a;f)\subset\overline{E}(b;g)$. If f is not a fractional linear transformation of g, then*

$$N(r,a;f\mid\leqslant 1)=0,\qquad N(r,b;f\mid\geqslant 2)+N(r,b;g\mid\geqslant 2)=S(r),\qquad \frac{f'(f-b)}{f(f-1)}\equiv\frac{g'(g-b)}{g(g-1)}$$

and f, g assume one of the following forms:

(i) $f=\frac{e^{3\gamma}-1}{e^{\gamma}-1}$ *and* $g=\frac{e^{-3\gamma}-1}{e^{-\gamma}-1}$ *with* $a=\frac{3}{4}$ *and* $b=3$;
(ii) $f=\frac{e^{3\gamma}-1}{e^{2\gamma}-1}$ *and* $g=\frac{e^{-3\gamma}-1}{e^{-2\gamma}-1}$ *with* $a=-3$ *and* $b=\frac{3}{2}$;
(iii) $f=\frac{e^{\gamma}-1}{e^{3\gamma}-1}$ *and* $g=\frac{e^{-\gamma}-1}{e^{-3\gamma}-1}$ *with* $a=\frac{4}{3}$ *and* $b=\frac{1}{3}$;
(iv) $f=\frac{e^{2\gamma}-1}{e^{3\gamma}-1}$ *and* $g=\frac{e^{-2\gamma}-1}{e^{-3\gamma}-1}$ *with* $a=-\frac{1}{3}$ *and* $b=\frac{2}{3}$;
(v) $f=\frac{e^{2\gamma}-1}{e^{-\gamma}-1}$ *and* $g=\frac{e^{-2\gamma}-1}{e^{\gamma}-1}$ *with* $a=\frac{1}{4}$ *and* $b=-2$;
(vi) $f=\frac{e^{\gamma}-1}{e^{-2\gamma}-1}$ *and* $g=\frac{e^{-\gamma}-1}{e^{2\gamma}-1}$ *with* $a=4$ *and* $b=-\frac{1}{2}$;
(vii) $f=\frac{e^{2\gamma}-1}{\lambda e^{\gamma}-1}$ *and* $g=\frac{e^{-2\gamma}-1}{\frac{1}{\lambda}e^{-\gamma}-1}$ *with* $\lambda^2\neq 1$, $a^2\lambda^2=4(a-1)$ *and* $b=2$;

(viii) $f = \frac{e^{\gamma}-1}{\lambda e^{2\gamma}-1}$ *and* $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{-2\gamma}-1}$ *with* $\lambda \neq 1$, $4a(1-a)\lambda = 1$ *and* $b = \frac{1}{2}$;

(ix) $f = \frac{e^{\gamma}-1}{\lambda e^{-\gamma}-1}$ *and* $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{\gamma}-1}$ *with* $\lambda \neq 1$, $(1-a)^2 + 4a\lambda = 0$ *and* $b = -1$, *where* γ *is a non-constant entire function.*

Moreover, if f *is a fractional linear transformation of* g *and* $f-a$ *and* $g-b$ *share* $(0,0)$, *then* f *and* g *satisfy exactly one of the following relations*: (i) $f \equiv g$; (ii) $fg \equiv 1$; (iii) $f \equiv \frac{a}{b}g$; (iv) $f + g \equiv 1$; (v) $f \equiv ag$; (vi) $f \equiv (1-a)g + a$; (vii) $f \equiv \frac{1-a}{1-b}g + \frac{b-a}{b-1}$; (viii) $f \equiv \frac{ag}{a-1+g}$; (ix) $f \equiv \frac{a(b-1)g}{(b-a)g+(a-1)b}$; (x) $f \equiv \frac{g}{g-1}$.

In the paper we present a two-fold improvement of Theorem B. In fact we reduce the weights of sharing of all the three values 0, 1, ∞ to some finite quantity and give specific forms of the functions. We now state the main result of the paper.

Theorem 1.1. *Let* f *and* g *be two distinct non-constant meromorphic functions sharing* $(a_1, 1)$, (a_2, m), (a_3, k), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$ *and* $(m-1)(mk-1) > (1+m)^2$. *Further let* $\overline{E}_{2)}(a; f) \subset \overline{E}(b; g)$ *for two complex numbers* $a, b \notin \{0, 1, \infty\}$. *Then* f *and* g *assume one of the following forms*:

(i) $f = ae^{\gamma}$ *and* $g = be^{-\gamma}$, *where* $ab = 1$;
(ii) $f = 1 + ae^{\gamma}$ *and* $g = 1 + (1 - \frac{1}{b})e^{-\gamma}$, *where* $ab = a + b$;
(iii) $f = \frac{a}{a+e^{\gamma}}$ *and* $g = \frac{e^{\gamma}}{1-b+e^{\gamma}}$, *where* $a + b = 1$;
(iv) $f = \frac{e^{\gamma}-a}{e^{\gamma}-1}$ *and* $g = \frac{e^{\gamma}-a}{ae^{\gamma}-a}$, *where* $\overline{E}(a; f) = \phi$;
(v) $f = \frac{be^{\gamma}-a}{be^{\gamma}-b}$ *and* $g = \frac{be^{\gamma}-a}{ae^{\gamma}-a}$, *where* $a \neq b$;
(vi) $f = \frac{a}{1-e^{\gamma}}$ *and* $g = \frac{ae^{\gamma}}{(1-a)(1-e^{\gamma})}$, *where* $\overline{E}(a; f) = \phi$;
(vii) $f = \frac{b-a}{(b-1)(1-e^{\gamma})}$ *and* $g = \frac{(b-a)e^{\gamma}}{(a-1)(1-e^{\gamma})}$, *where* $a \neq b$;
(viii) $f = a + e^{\gamma}$ *and* $g = (1-a)(1 + \frac{a}{e^{\gamma}})$, *where* $\overline{E}(a; f) = \phi$;
(ix) $f = e^{\gamma} - \frac{a(b-1)}{a-b}$ *and* $g = \frac{b(a-1)}{a-b}\{1 - \frac{a(b-1)}{(a-b)e^{\gamma}}\}$, *where* $a \neq b$;
(x) $f = \frac{e^{3\gamma}-1}{e^{\gamma}-1}$ *and* $g = \frac{e^{-3\gamma}-1}{e^{-\gamma}-1}$ *with* $a = \frac{3}{4}$ *and* $b = 3$;
(xi) $f = \frac{e^{3\gamma}-1}{e^{2\gamma}-1}$ *and* $g = \frac{e^{-3\gamma}-1}{e^{-2\gamma}-1}$ *with* $a = -3$ *and* $b = \frac{3}{2}$;
(xii) $f = \frac{e^{\gamma}-1}{e^{3\gamma}-1}$ *and* $g = \frac{e^{-\gamma}-1}{e^{-3\gamma}-1}$ *with* $a = \frac{4}{3}$ *and* $b = \frac{1}{3}$;
(xiii) $f = \frac{e^{2\gamma}-1}{e^{3\gamma}-1}$ *and* $g = \frac{e^{-2\gamma}-1}{e^{-3\gamma}-1}$ *with* $a = -\frac{1}{3}$ *and* $b = \frac{2}{3}$;
(xiv) $f = \frac{e^{2\gamma}-1}{e^{-\gamma}-1}$ *and* $g = \frac{e^{-2\gamma}-1}{e^{\gamma}-1}$ *with* $a = \frac{1}{4}$ *and* $b = -2$;
(xv) $f = \frac{e^{\gamma}-1}{e^{-2\gamma}-1}$ *and* $g = \frac{e^{-\gamma}-1}{e^{2\gamma}-1}$ *with* $a = 4$ *and* $b = -\frac{1}{2}$;
(xvi) $f = \frac{e^{2\gamma}-1}{\lambda e^{\gamma}-1}$ *and* $g = \frac{e^{-2\gamma}-1}{\frac{1}{\lambda}e^{-\gamma}-1}$ *with* $\lambda^2 \neq 1$, $a^2\lambda^2 = 4(a-1)$ *and* $b = 2$;
(xvii) $f = \frac{e^{\gamma}-1}{\lambda e^{2\gamma}-1}$ *and* $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{-2\gamma}-1}$ *with* $\lambda \neq 1$, $4a(1-a)\lambda = 1$ *and* $b = \frac{1}{2}$;
(xviii) $f = \frac{e^{\gamma}-1}{\lambda e^{-\gamma}-1}$ *and* $g = \frac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{\gamma}-1}$ *with* $\lambda \neq 1$, $(1-a)^2 + 4a\lambda = 0$ *and* $b = -1$, *where* γ *is a non-constant entire function.*

Following theorem is a consequence of Theorem 1.1 and improves Theorem A and Theorem 1.1 of [14].

Theorem 1.2. *Let* f *and* g *be two distinct non-constant meromorphic functions sharing* $(a_1, 1)$, (a_2, m), (a_3, k), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$ *and* $(m-1)(mk-1) > (1+m)^2$. *If for two complex numbers* $a, b \notin \{0, 1, \infty\}$, $\overline{E}_{2)}(a; f) \subset \overline{E}(b; g)$ *and* $\overline{E}_{1)}(b; g) \subset \overline{E}(a; f)$, *then* f, g *share* $(0, \infty)$, $(1, \infty)$, (∞, ∞) *and* $f-a$, $g-b$ *share* $(0, \infty)$. *Also there exists a non-constant entire function* γ *such that* f *and* g *assume one of the following forms*:

(i) $f = ae^{\gamma}$ *and* $g = be^{-\gamma}$, *where* $ab = 1$.
(ii) $f = 1 + ae^{\gamma}$ *and* $g = 1 + (1 - \frac{1}{b})e^{-\gamma}$, *where* $ab = a + b$.
(iii) $f = \frac{a}{a+e^{\gamma}}$ *and* $g = \frac{e^{\gamma}}{1-b+e^{\gamma}}$, *where* $a + b = 1$.
(iv) $f = \frac{e^{\gamma}-a}{e^{\gamma}-1}$ *and* $g = \frac{be^{\gamma}-1}{e^{\gamma}-1}$, *where* $ab = 1$.
(v) $f = \frac{be^{\gamma}-a}{be^{\gamma}-b}$ *and* $g = \frac{be^{\gamma}-a}{ae^{\gamma}-a}$, *where* $a \neq b$.
(vi) $f = \frac{a}{1-e^{\gamma}}$ *and* $g = \frac{be^{\gamma}}{e^{\gamma}-1}$, *where* $ab = a + b$.
(vii) $f = \frac{b-a}{(b-1)(1-e^{\gamma})}$ *and* $g = \frac{(b-a)e^{\gamma}}{(a-1)(1-e^{\gamma})}$, *where* $a \neq b$.
(viii) $f = a + e^{\gamma}$ *and* $g = b(1 + \frac{1-b}{e^{\gamma}})$, *where* $a + b = 1$.
(ix) $f = e^{\gamma} - \frac{a(b-1)}{a-b}$ *and* $g = \frac{b(a-1)}{a-b}\{1 - \frac{a(b-1)}{(a-b)e^{\gamma}}\}$, *where* $a \neq b$.

Following example shows that the condition $\overline{E}_{2)}(a; f) \subset \overline{E}(b; g)$ and $\overline{E}_{1)}(b; g) \subset \overline{E}(a; f)$ cannot be further relaxed in Theorem 1.2.

Example 1.2. Let $f = e^{2z} + e^{z} + 1$ and $g = e^{-2z} + e^{-z} + 1$. Then $\overline{E}_{1)}(\frac{3}{4}; f) = \overline{E}_{1)}(\frac{3}{4}; g)$ and $\overline{E}_{2)}(\frac{3}{4}; f) \subset \overline{E}(3; g)$ but $\overline{E}_{1)}(3; g) \not\subset \overline{E}(\frac{3}{4}; f)$. Also f and g share $(0, \infty)$, $(1, \infty)$, (∞, ∞) but assume none of the forms as given in Theorem 1.2.

Though for the standard definitions and notations of the value distribution theory we refer to [6], we now explain a notation.

Definition 1.4. For two meromorphic functions f and g and for $a, b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid g = b)$ the counting function of those a-points of f (counted with multiplicities) which are b-points of g.

2. Lemmas

In this section we present some lemmas which are required in the sequel.

Lemma 2.1. *(See [4].) If f, g share $(0,0)$, $(1,0)$, $(\infty,0)$, then $T(r, f) \leqslant 3T(r, g) + S(r, f)$ and $T(r, g) \leqslant 3T(r, f) + S(r, g)$.*

This shows that $S(r, f) = S(r, g)$ and we denote them by $S(r)$.

Lemma 2.2. *(See [9].) Let f, g share $(0,1)$, $(1,m)$, (∞,k) and $f \not\equiv g$, where $(m-1)(mk-1) > (1+m)^2$. Then for $a = 0, 1, \infty$*

$$\overline{N}(r, a; f \mid \geqslant 2) + \overline{N}(r, a; g \mid \geqslant 2) = S(r).$$

Lemma 2.3. *(See [10,12].) Let f, g share $(0,1)$, $(1,m)$, (∞,k) and $f \not\equiv g$, where $(m-1)(mk-1) > (1+m)^2$. If f is not a bilinear transformation of g, then each of the following holds*:

(i) $T(r, f) + T(r, g) = N(r, 0; f \mid \leqslant 1) + N(r, 1; f \mid \leqslant 1) + N(r, \infty; f \mid \leqslant 1) + N_0(r) + S(r)$,
(ii) $T(r, f) = N(r, 0; g' \mid \leqslant 1) + N_0(r) + S(r)$,
(iii) $T(r, g) = N(r, 0; f' \mid \leqslant 1) + N_0(r) + S(r)$,
(iv) $N_1(r) = S(r)$,
(v) $N_0(r, 0; g' \mid \geqslant 2) = S(r)$,
(vi) $N_0(r, 0; f' \mid \geqslant 2) = S(r)$,
(vii) $\overline{N}(r, 0; g' \mid \geqslant 2) = S(r)$,
(viii) $\overline{N}(r, 0; f' \mid \geqslant 2) = S(r)$,
(ix) $N(r, 0; f - g \mid \geqslant 2) = S(r)$,
(x) $N(r, 0; f - g \mid f = \infty) = S(r)$,

where $N_0(r)$ ($N_1(r)$) denotes the counting function of those simple (multiple) zeros of $f-g$ which are not the zeros of $f(f-1)$ and $\frac{1}{f}$; also $N_0(r,0;g'\mid\geqslant 2)$ ($N_0(r,0;f'\mid\geqslant 2)$) is the counting function of those multiple zeros of $g'(f')$ which are not the zeros of $f(f-1)$.

Lemma 2.4. *Let f, g share $(0,1)$, $(1,m)$, (∞,k) and $f\not\equiv g$, where $(m-1)(mk-1)>(1+m)^2$. If f is not a bilinear transformation of g, then for a complex number $a\notin\{0,1,\infty\}$ each of the following holds:*

(i) $N(r,a;f\mid\geqslant 3)+N(r,a;g\mid\geqslant 3)=S(r)$,
(ii) $T(r,f)=N(r,a;f\mid\leqslant 2)+S(r)$,
(iii) $T(r,g)=N(r,a;g\mid\leqslant 2)+S(r)$.

Proof. By (v) and (vi) of Lemma 2.3 we get

$$N(r,a;f\mid\geqslant 3)+N(r,a;g\mid\geqslant 3)\leqslant 2N_0(r,0;f'\mid\geqslant 2)+2N_0(r,0;g'\mid\geqslant 2)=S(r),$$

which is (i).

Again by the second fundamental theorem, Lemma 2.2, (i), (iii) and (vi) of Lemma 2.3 we get

$$\begin{aligned}2T(r,f)&\leqslant N(r,a;f)+N(r,0;f\mid\leqslant 1)+N(r,1;f\mid\leqslant 1)+N(r,\infty;f\mid\leqslant 1)-N_0(r,0;f'\mid\leqslant 1)+S(r)\\&=N(r,a;f)+T(r,f)+T(r,g)-N_0(r)-N_0(r,0;f'\mid\leqslant 1)+S(r)\\&=N(r,a;f)+T(r,f)+N(r,0;f'\mid\leqslant 1)-N_0(r,0;f'\mid\leqslant 1)+S(r),\end{aligned}\tag{2.1}$$

where $N_0(r,0;f'\mid\leqslant 1)$ denotes the counting function of those simple zeros of f' which are not the zeros of $f(f-1)$.

Now by Lemma 2.2 we get

$$\begin{aligned}N(r,0;f'\mid\leqslant 1)&\leqslant N_0(r,0;f'\mid\leqslant 1)+\overline{N}(r,0;f\mid\geqslant 2)+\overline{N}(r,1;f\mid\geqslant 2)\\&=N_0(r,0;f'\mid\leqslant 1)+S(r)\end{aligned}$$

and so $N(r,0;f'\mid\leqslant 1)=N_0(r,0;f'\mid\leqslant 1)+S(r)$. Hence from (2.1) and (i) of this lemma we get

$$T(r,f)=N(r,a;f\mid\leqslant 2)+S(r),$$

which is (ii). Similarly we can prove (iii). This proves the lemma. □

Lemma 2.5. *(See [11].) Let f, g share $(0,1)$, $(1,m)$, (∞,k) and $f\not\equiv g$, where $(m-1)(mk-1)>(1+m)^2$. If $\alpha=\frac{f-1}{g-1}$ and $\beta=\frac{g}{f}$, then $\overline{N}(r,a;\alpha)+\overline{N}(r,a;\beta)=S(r)$ for $a=0,\infty$.*

Lemma 2.6. *Let f and g be distinct meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1)>(1+m)^2$. Then $T(r,\frac{\alpha^{(p)}}{\alpha})+T(r,\frac{\beta^{(p)}}{\beta})=S(r)$, where p is a positive integer and α, β are defined as in Lemma 2.5.*

Proof. Since $\alpha=\frac{f-1}{g-1}$ and $\beta=\frac{g}{f}$, we get $T(r,\alpha)\leqslant T(r,f)+T(r,g)+O(1)$ and $T(r,\beta)\leqslant T(r,f)+T(r,g)+O(1)$.

So by Lemma 2.1 we see that $S(r,\alpha)$ and $S(r,\beta)$ are replaceable by $S(r)$. Now by Lemma 2.5 we get

$$\begin{aligned}T\left(r,\frac{\alpha^{(p)}}{\alpha}\right)&=N\left(r,\frac{\alpha^{(p)}}{\alpha}\right)+m\left(r,\frac{\alpha^{(p)}}{\alpha}\right)\\&\leqslant p\overline{N}(r,0;\alpha)+p\overline{N}(r,\infty;\alpha)+S(r)\\&=S(r).\end{aligned}$$

Similarly we can prove $T(r,\frac{\beta^{(p)}}{\beta})=S(r)$. This proves the lemma. □

Lemma 2.7. *(See [13].) Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1)>(1+m)^2$. If $N_0(r)+N_1(r)\geqslant\lambda T(r,f)+S(r)$ for some $\lambda>\frac{1}{2}$, then f is a bilinear transformation of g and $N_0(r)+N_1(r)=T(r,f)+S(r)=T(r,g)+S(r)$.*

Lemma 2.8. *(See [11].) Let f and g be distinct meromorphic functions sharing* $(0,0)$, $(1,0)$ *and* $(\infty,0)$. *If f is a bilinear transformation of g, then f and g satisfy one of the following*: (i) $fg\equiv 1$, (ii) $(f-1)(g-1)\equiv 1$, (iii) $f+g\equiv 1$, (iv) $f\equiv cg$, (v) $f-1\equiv c(g-1)$, (vi) $\{(c-1)f+1\}\{(c-1)g-c\}+c\equiv 0$, *where c $(\neq 0,1,\infty)$ is a constant.*

Lemma 2.9. *Let f and g be two distinct non-constant meromorphic functions sharing* $(0,0)$, $(1,0)$, $(\infty,0)$. *Further suppose that f is a bilinear transformation of g and $\overline{E}_{1)}(a;f)\subset\overline{E}(b;g)$, where $a,b\notin\{0,1,\infty\}$. Then there exists a non-constant entire function γ such that f and g are one of the following forms*:

(i) $f=ae^{\gamma}$ *and* $g=be^{-\gamma}$, *where* $ab=1$;
(ii) $f=1+ae^{\gamma}$ *and* $g=1+(1-\frac{1}{b})e^{-\gamma}$, *where* $ab=a+b$;
(iii) $f=\frac{a}{a+e^{\gamma}}$ *and* $g=\frac{e^{\gamma}}{1-b+e^{\gamma}}$, *where* $a+b=1$;
(iv) $f=\frac{e^{\gamma}-a}{e^{\gamma}-1}$ *and* $g=\frac{e^{\gamma}-a}{ae^{\gamma}-a}$, *where* $\overline{E}(a;f)=\phi$;
(v) $f=\frac{be^{\gamma}-a}{be^{\gamma}-b}$ *and* $g=\frac{be^{\gamma}-a}{ae^{\gamma}-a}$, *where* $a\neq b$;
(vi) $f=\frac{a}{1-e^{\gamma}}$ *and* $g=\frac{ae^{\gamma}}{(1-a)(1-e^{\gamma})}$, *where* $\overline{E}(a;f)=\phi$;
(vii) $f=\frac{b-a}{(b-1)(1-e^{\gamma})}$ *and* $g=\frac{(b-a)e^{\gamma}}{(a-1)(1-e^{\gamma})}$, *where* $a\neq b$;
(viii) $f=a+e^{\gamma}$ *and* $g=(1-a)(1+\frac{a}{e^{\gamma}})$, *where* $\overline{E}(a;f)=\phi$;
(ix) $f=e^{\gamma}-\frac{a(b-1)}{a-b}$ *and* $g=\frac{b(a-1)}{a-b}\{1-\frac{a(b-1)}{(a-b)e^{\gamma}}\}$, *where* $a\neq b$.

Proof. Clearly f and g satisfy one of the six relations of Lemma 2.8. Let $fg\equiv 1$. Then f and g do not assume the values 0 and ∞. Hence there exists a non-constant entire function γ such that $f=ae^{\gamma}$ and $g=\frac{1}{a}e^{-\gamma}$. If $f-a$ has no simple zero, then $\Theta(a;f)\geqslant\frac{1}{2}$, which is impossible. So $\overline{E}_{1)}(a;f)\neq\phi$ and $\overline{E}_{1)}(a;f)\subset\overline{E}(b;g)$ implies $f=ae^{\gamma}$ and $g=be^{-\gamma}$, where $ab=1$. This is the possibility (i).

Let $(f-1)(g-1)\equiv 1$. Then f and g do not assume the values 1 and ∞. Hence there exists a non-constant entire function γ such that $f=1+ae^{\gamma}$ and $g=1+\frac{1}{a}e^{-\gamma}$. Since $\overline{E}_{1)}(a;f)\neq\phi$ and $\overline{E}_{1)}(a;f)\subset\overline{E}(b;g)$, we get $ab=a+b$. Therefore $f=1+ae^{\gamma}$ and $g=1+(1-\frac{1}{b})e^{-\gamma}$, where $ab=a+b$. This is the possibility (ii).

Let $f+g\equiv 1$. Then f and g do not assume the values 0 and 1. So there exists a non-constant entire function γ such that $f=\frac{a}{a+e^{\gamma}}$ and $g=\frac{e^{\gamma}}{a+e^{\gamma}}$. Since $\overline{E}_{1)}(a;f)\neq\phi$ and $\overline{E}_{1)}(a;f)\subset\overline{E}(b;g)$, we get $a+b=1$. Therefore $f=\frac{a}{a+e^{\gamma}}$ and $g=\frac{e^{\gamma}}{1-b+e^{\gamma}}$, where $a+b=1$. This is the possibility (iii).

Let $f\equiv cg$. Then f does not assume the values 1 and c. Hence there exists a non-constant entire function γ such that $f=\frac{e^{\gamma}-c}{e^{\gamma}-1}$ and $g=\frac{e^{\gamma}-c}{ce^{\gamma}-c}$. If $\overline{E}_{1)}(a;f)=\phi$, then $\Theta(a;f)\geqslant\frac{1}{2}$ and so $c=a$. Hence $f=\frac{e^{\gamma}-a}{e^{\gamma}-1}$ and $g=\frac{e^{\gamma}-a}{ae^{\gamma}-a}$, where $\overline{E}(a;f)=\phi$. This is the possibility (iv). If $\overline{E}_{1)}(a;f)\neq\phi$, then $\overline{E}_{1)}(a;f)\subset\overline{E}(b;g)$ implies $c=\frac{a}{b}$. Since $c\neq 1$, we get $a\neq b$. Therefore from above we obtain $f=\frac{be^{\gamma}-a}{be^{\gamma}-b}$ and $g=\frac{be^{\gamma}-a}{ae^{\gamma}-a}$, where $a\neq b$. This is the possibility (v).

Let $f-1\equiv c(g-1)$. Then f does not assume the values 0 and $1-c$. So there exists a non-constant entire function γ such that $f=\frac{1-c}{1-e^{\gamma}}$ and $g=\frac{e^{\gamma}(1-c)}{c(1-e^{\gamma})}$. If $\overline{E}_{1)}(a;f)=\phi$, then $\Theta(a;f)\geqslant\frac{1}{2}$ and so $a=1-c$. Hence $f=\frac{a}{1-e^{\gamma}}$ and $g=\frac{ae^{\gamma}}{(1-a)(1-e^{\gamma})}$, where $\overline{E}(a;f)=\phi$. This is the possibility (vi). If $\overline{E}_{1)}(a;f)\neq\phi$ then $\overline{E}_{1)}(a;f)\subset\overline{E}(b;g)$ implies that $c=\frac{a-1}{b-1}$. Since $c\neq 1$, we get $a\neq b$. Therefore $f=\frac{b-a}{(b-1)(1-e^{\gamma})}$ and $g=\frac{(b-a)e^{\gamma}}{(a-1)(1-e^{\gamma})}$, where $a\neq b$. This is the possibility (vii).

Let $\{(c-1)f+1\}\{(c-1)g-c\}+c\equiv 0$. Then f does not assume the values ∞ and $\frac{1}{1-c}$. So there exists a non-constant entire function γ such that $f=\frac{1}{1-c}+e^{\gamma}$ and $g=\frac{c}{c-1}\{1+\frac{1}{(1-c)e^{\gamma}}\}$. If $\overline{E}_{1)}(a;f)=\phi$, then $\Theta(a;f)\geqslant\frac{1}{2}$ and so $a=\frac{1}{1-c}$. Hence $f=a+e^{\gamma}$ and $g=(1-a)(1+\frac{a}{e^{\gamma}})$, where $\overline{E}(a;f)=\phi$. This is the possibility (viii). If $\overline{E}_{1)}(a;f)\neq\phi$, then $\overline{E}_{1)}(a;f)\subset\overline{E}(b;g)$ implies that $c=\frac{b(a-1)}{a(b-1)}$. Since $c\neq 1$, we get $a\neq b$. Therefore $f=e^{\gamma}-\frac{a(b-1)}{a-b}$ and $g=\frac{b(a-1)}{a-b}\{1-\frac{a(b-1)}{(a-b)e^{\gamma}}\}$, where $a\neq b$. This is the possibility (ix). This proves the lemma. □

Lemma 2.10. *(See [15], [17, p. 28].) Let f be a non-constant meromorphic function and $R(f)=\frac{P(f)}{Q(f)}$, where $P(f)=\sum_{i=0}^{p}a_if^i$ and $Q(f)=\sum_{j=0}^{q}b_jf^j$ are two mutually prime polynomials in f of degree p and q, respectively.*

If $T(r,a_i)=S(r,f)$ and $T(r,b_j)=S(r,f)$ for $i=1,2,\dots,p$ and $j=1,2,\dots q$, then

$$T\big(r,R(f)\big)=\max\{p,q\}T(r,f)+S(r,f).$$

3. Proofs of theorems

Proof of Theorem 1.1. First we suppose that $a_1=0$, $a_2=1$ and $a_3=\infty$. We now consider the following cases.

Case 1. Let $a=b$. We put

$$\chi=\frac{f'(f-a)}{f(f-1)}-\frac{g'(g-a)}{g(g-1)}.$$

Suppose that $\chi\not\equiv 0$. Since $\chi=a\frac{\beta'}{\beta}+(1-a)\frac{\alpha'}{\alpha}$, by Lemma 2.6 we get $T(r,\chi)=S(r)$. From the given condition we see that $N(r,a;f\mid\leqslant 2)\leqslant 2N(r,0;\chi)=S(r)$, which contradicts Lemma 2.4(ii). Therefore $\chi\equiv 0$ and so

$$\frac{f'(f-a)}{f(f-1)}\equiv\frac{g'(g-a)}{g(g-1)}. \tag{3.1}$$

From (3.1) it follows that f, g share $(0,\infty)$, $(1,\infty)$ and (∞,∞). Also from (3.1) we see that a double zero of $f-a$ is a zero of $\frac{\beta'}{\beta}$. Therefore by Lemma 2.6 we get

$$N(r,a;f\mid=2)\leqslant 2N\left(r,0;\frac{\beta'}{\beta}\right)=S(r),$$

where $N(r,a;f\mid=2)$ denotes the counting function of double zeros of $f-a$, counted with multiplicities. Hence by Lemma 2.4(ii) we obtain

$$N(r,a;f\mid\leqslant 1)=T(r,f)+S(r).$$

Therefore by Theorem B f is a bilinear transformation of g. So by Lemma 2.9 f and g assume one of the forms (i)–(iv), (vi), (viii).

Case 2. Let $a\neq b$. If f is a bilinear transformation of g, then by Lemma 2.9 f and g assume one of the forms (i)–(ix). So we now suppose that f is not a bilinear transformation of g. Following two subcases come up for consideration.

Subcase (i). Let $N(r,a;f\mid\geqslant 2)\neq S(r)$. We define

$$\psi=\frac{f'(f-b)}{f(f-1)}-\frac{g'(g-b)}{g(g-1)}.$$

Since a double zero of $f-a$ is a zero of $g-b$, if $\psi\not\equiv 0$, then by Lemmas 2.4(i) and 2.6 we get

$$N(r,a;f\mid\geqslant 2)\leqslant 2N(r,0;\psi)+S(r)=S(r),$$

which is impossible. Therefore $\psi\equiv 0$ and so

$$\frac{f'(f-b)}{f(f-1)}\equiv\frac{g'(g-b)}{g(g-1)}. \tag{3.2}$$

From (3.2) we see that f and g share $(0,\infty)$, $(1,\infty)$ and (∞,∞). Hence by Theorem B f and g assume one of the forms (x)–(xviii).

Subcase (ii). Let $N(r,a;f\mid\geqslant 2)=S(r)$. Since f is not a bilinear transformation of g, we see that α, β and $\alpha\beta$ are non-constant. Also we note that $f=\frac{1-\alpha}{1-\alpha\beta}$ and $g=\frac{(1-\alpha)\beta}{1-\alpha\beta}$.

We put $F=(f-a)(1-\alpha\beta)=a\alpha\beta-\alpha+1-a$ and $w=\frac{F'}{F}$. Since $1-\alpha\beta=\frac{g-f}{f(g-1)}$, we get $F=(f-a)\frac{g-f}{f(g-1)}$. Since by Lemma 2.5 $\overline{N}(r,\infty;F)=S(r)$ and w has only simple poles (if there is any), we get

$$T(r,w)=m(r,w)+N(r,w)=\overline{N}(r,0;F)+S(r). \tag{3.3}$$

By (ix), (x) of Lemma 2.3 and Lemma 2.2 we get

$$\begin{aligned}\overline{N}(r,0;F\mid\geqslant 2)&\leqslant N(r,a;f\mid\geqslant 2)+N(r,0;f-g\mid\geqslant 2)+\overline{N}(r,\infty;f\mid\geqslant 2)+N(r,0;f-g\mid f=\infty)\\&=S(r).\end{aligned}\tag{3.4}$$

Hence from (3.3) and (3.4) we get in view of (ix) of Lemma 2.3

$$T(r,w)=N(r,0;F\mid\leqslant 1)+S(r)=N(r,a;f\mid\leqslant 1)+N_0(r)+N_2(r)+S(r),\tag{3.5}$$

where $N_2(r)$ is the counting function of those simple poles of f which are non-zero regular points of $f-g$.

From the definitions of α and β we get

$$\left\{g-\frac{\alpha'\beta}{(\alpha\beta)'}\right\}\left(\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta}\right)\equiv\frac{f'(g-f)}{f(f-1)}.\tag{3.6}$$

From (3.6) we see that a simple pole of f which is a non-zero regular point of $f-g$ is a regular point of $\{g-\frac{\alpha'\beta}{(\alpha\beta)'}\}(\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta})$. Hence it is either a pole of $\frac{\alpha'\beta}{(\alpha\beta)'}$ or a zero of $\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta}$. Therefore by Lemma 2.6 and the first fundamental theorem we get

$$\begin{aligned}N_2(r)&\leqslant T\left(r,\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta}\right)+T\left(r,\frac{\alpha'\beta}{(\alpha\beta)'}\right)\\&\leqslant T\left(r,\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta}\right)+T\left(r,\frac{1}{1+\frac{\alpha\beta'}{\alpha'\beta}}\right)\\&\leqslant 2T\left(r,\frac{\alpha'}{\alpha}\right)+2T\left(r,\frac{\beta'}{\beta}\right)+O(1)\\&=S(r).\end{aligned}$$

So from (3.5) we get

$$T(r,w)=N(r,a;f\mid\leqslant 1)+N_0(r)+S(r).\tag{3.7}$$

By (ii) of Lemma 2.4 we get from (3.7)

$$T(r,w)=T(r,f)+N_0(r)+S(r).\tag{3.8}$$

Let

$$\begin{aligned}\tau_1&=\frac{a-1}{b-1}(\xi-b\delta),\\\tau_2&=\frac{1}{2}\cdot\frac{a-1}{b-1}\left\{\xi'+\xi^2-b\left(\delta'+\delta^2\right)\right\}\end{aligned}$$

and

$$\tau_3=\frac{1}{6}\cdot\frac{a-1}{b-1}\left\{\xi''+3\xi\xi'+\xi^3-b\left(\delta''+3\delta\delta'+\delta^3\right)\right\},$$

where $\xi=\frac{\alpha'}{\alpha}$ and $\delta=\frac{\alpha'}{\alpha}+\frac{\beta'}{\beta}$. By Lemma 2.6 we see that $T(r,\xi)=S(r)$ and $T(r,\delta)=S(r)$.

If $\tau_1\equiv 0$, from (3.6) we get

$$(g-b)\delta\equiv\frac{f'(g-f)}{f(f-1)}.\tag{3.9}$$

Since $\overline{E}_{2)}(a;f)\subset\overline{E}(b;g)$, it follows from (3.9) that a simple zero of $f-a$, which is neither a zero nor a pole of δ, is a zero of $g-b$ and so is a zero of f'. Hence $N(r,a;f\mid\leqslant 1)=S(r)$, which contradicts (ii) of Lemma 2.4. Therefore $\tau_1\not\equiv 0$.

Let z_0 be a simple zero of $f-a$ and $\tau_1(z_0)\neq 0$. Then $g(z_0)=b$ and so $\alpha(z_0)=\frac{a-1}{b-1}$ and $\beta(z_0)=\frac{b}{a}$. Expanding F around z_0 in Taylor's series we get

$$-F(z)=\tau_1(z_0)(z-z_0)+\tau_2(z_0)(z-z_0)^2+\tau_3(z_0)(z-z_0)^3+O\left((z-z_0)^4\right).$$

Hence in some neighbourhood of z_0 we obtain

$$w(z)=\frac{1}{z-z_0}+\frac{B(z_0)}{2}+C(z_0)(z-z_0)+O\big((z-z_0)^2\big), \tag{3.10}$$

where $B=\frac{2\tau_2}{\tau_1}$ and $C=\frac{2\tau_3}{\tau_1}-(\frac{\tau_2}{\tau_1})^2$.

We put

$$H=w'+w^2-Bw-A, \tag{3.11}$$

where $A=3C-\frac{B^2}{4}-B'$.

Clearly $T(r,A)+T(r,B)+T(r,C)=S(r)$ and since $w=\frac{F'}{F}$ and $F=(f-a)\frac{g-f}{f(g-1)}$, we get by Lemma 2.1 and (3.8) that $S(r,w)=S(r)$.

Let $H\not\equiv 0$. Then it is easy to see that z_0 is a zero of H. So

$$\begin{aligned}N(r,a;f\mid\leqslant 1)&\leqslant N(r,0;H)+S(r)\\&\leqslant T(r,H)+S(r)\\&=N(r,H)+S(r).\end{aligned} \tag{3.12}$$

By Lemma 2.4(ii) and (3.12) we get

$$T(r,f)\leqslant N(r,H)+S(r). \tag{3.13}$$

Let z_1 be a pole of F. Then z_1 is a simple pole of w. So if z_1 is not a pole of A and B, then z_1 is at most a double pole of H. Hence by Lemma 2.5 we get

$$N(r,\infty;H\mid F=\infty)\leqslant 2\overline{N}(r,\infty;F)+S(r)=S(r). \tag{3.14}$$

Let z_2 be a multiple zero of F. Then z_2 is a simple pole of w. So if z_2 is not a pole of A and B, then z_2 is a pole of H of multiplicity at most two. Hence by (3.4)

$$N(r,\infty;H\mid F=0,\geqslant 2)\leqslant 2\overline{N}(r,0;F\mid\geqslant 2)+S(r)=S(r), \tag{3.15}$$

where $N(r,\infty;H\mid F=0,\geqslant 2)$ denotes the counting function of those poles of H which are multiple zeros of F.

Let z_3 be a simple zero of F which is not a pole of A and B. Then in some neighbourhood of z_3 we get $F(z)=(z-z_3)h(z)$, where h is analytic at z_3 and $h(z_3)\neq 0$. Hence in some neightbourhood of z_3 we obtain

$$H(z)=\left(\frac{2h'}{h}-B\right)\frac{1}{z-z_3}+h_1,$$

where $h_1=(\frac{h'}{h})'+(\frac{h'}{h})^2-\frac{Bh'}{h}-A$.

This shows that z_3 is at most a simple pole of H. Since a simple zero of $f-a$ is a zero of H and $N(r,0;F\mid f=t)\leqslant N(r,0;f-g\mid\geqslant 2)$ for $t=0,1$ and $F=(f-a)\frac{g-f}{f(g-1)}$, we get from (3.14) and (3.15) in view of (ix) of Lemma 2.3

$$\begin{aligned}N(r,H)&=N(r,\infty;H\mid F=\infty)+N(r,\infty;H\mid F=0)+S(r)\\&\leqslant N(r,0;F\mid\leqslant 1)-N(r,a;f\mid\leqslant 1)+S(r)\\&=N_0(r)+N_2(r)+S(r)\\&=N_0(r)+S(r).\end{aligned} \tag{3.16}$$

From (3.13) and (3.16) we obtain $T(r,f)\leqslant N_0(r)+S(r)$, which by (iv) of Lemma 2.3 contradicts Lemma 2.7. Therefore $H\equiv 0$ and so

$$w'+w^2-Bw-A\equiv 0. \tag{3.17}$$

From (3.17) we get $\frac{w'}{w}\equiv\frac{A}{w}+B-w$ and so $F''\equiv AF+BF'$. Since $F'=a(\alpha\beta)'-\alpha'$ and $F''=a(\alpha\beta)''-\alpha''$, we get

$$a\alpha\beta\left[\frac{(\alpha\beta)''}{\alpha\beta}-B\frac{(\alpha\beta)'}{\alpha\beta}\right]+\alpha\left[B\frac{\alpha'}{\alpha}-\frac{\alpha''}{\alpha}\right]\equiv A(f-a)(1-\alpha\beta). \tag{3.18}$$

Since $\alpha\beta = \frac{g(f-1)}{f(g-1)}$ and $\alpha = \frac{f-1}{g-1}$, we get from (3.18)

$$Kg(f-1) + Lf(f-1) \equiv A(f-a)(g-f),$$

where $K = \frac{(\alpha\beta)''}{\alpha\beta} - B\frac{(\alpha\beta)'}{\alpha\beta}$ and $L = B\frac{\alpha'}{\alpha} - \frac{\alpha''}{\alpha}$. By Lemma 2.6 we see that $T(r,K) = S(r)$ and $T(r,L) = S(r)$. Therefore

$$f[B_0 f - B_0 - B_2] + g\big[(B_1 + B_2)f - B_1\big] \equiv 0, \tag{3.19}$$

where $B_0 = L + A$, $B_1 = K - aA$ and $B_2 = (a-1)A$.

We now verify that if $K \equiv 0$ and $L \equiv 0$ then $B \not\equiv 0$. For, otherwise we get $(\alpha\beta)'' \equiv \alpha'' \equiv 0$. Hence $\alpha = p_1 z + q_1$ and $\alpha\beta = p_2 z + q_2$, where p_1, p_2, q_1, q_2 are constants. Also $p_1 \neq 0$ and $p_2 \neq 0$ because α and $\alpha\beta$ are non-constant. Further

$$f = \frac{p_1 z + q_1 - 1}{p_2 z + q_2 - 1} \quad \text{and} \quad g = \frac{(p_2 z + q_2)(p_1 z + q_1 - 1)}{(p_1 z + q_1)(p_2 z + q_2 - 1)}.$$

Since f, g share $(0,1)$ and (∞, k), we see that $\frac{p_2 z + q_2}{p_1 z + q_1}$ is a constant and so g becomes a bilinear transformation of f, which is a contradiction.

Since $\overline{E}_{2)}(a; f) \subset \overline{E}(b; g)$, we get $N(r, a; f \mid \leqslant 1) \leqslant N(r, \frac{b}{a}; \beta)$ and $N(r, a; f \mid \leqslant 1) \leqslant N(r, \frac{a-1}{b-1}; \alpha)$ and so by (ii)–(iii) of Lemma 2.4 we obtain

$$T(r,f) \leqslant T(r,\beta) + S(r) \tag{3.20}$$

and

$$T(r,f) \leqslant T(r,\alpha) + S(r). \tag{3.21}$$

If $(B_1 + B_2)f - B_1 \equiv 0$, then $B_1 \equiv B_2 \equiv 0$ because otherwise $T(r,f) = S(r)$, a contradiction. So from (3.19) we get $B_0 \equiv 0$. Therefore $K \equiv L \equiv 0$ and so $B \not\equiv 0$. Hence $\frac{(\alpha\beta)''}{(\alpha\beta)'} \equiv \frac{\alpha''}{\alpha'}$. On integration we obtain

$$(\alpha\beta)' \equiv c\alpha', \tag{3.22}$$

where $c(\neq 0)$ is a constant.

From (3.22) we get $\frac{c}{\beta} = 1 + \frac{\alpha}{\alpha'} \cdot \frac{\beta'}{\beta}$ and so by Lemma 2.6 we obtain $T(r,\beta) = S(r)$, which contradicts (3.20). Therefore $(B_1 + B_2)f - B_1 \not\equiv 0$ and so from (3.19) we get

$$g \equiv f\frac{A_1 f + A_2}{A_3 f + A_4}, \tag{3.23}$$

where $A_1 = B_0$, $A_2 = -B_0 - B_2$, $A_3 = B_1 + B_2$, $A_4 = -B_1$ and $T(r, A_j) = S(r)$ for $j = 1,2,3,4$.

Let $A_1 \equiv 0$. If $A_3 \equiv 0$, then from (3.23) we get $\beta = \frac{A_2}{A_4}$, which contradicts (3.20) and if $A_4 \equiv 0$, then from (3.23) we get $g = \frac{A_2}{A_3}$, which implies $T(r,g) = S(r)$, a contradiction. So $A_3 \not\equiv 0$ and $A_4 \not\equiv 0$. Also $A_2 \not\equiv 0$ because g is non-constant. If $A_3 + A_4 \equiv 0$, then $g = \frac{A_2 f}{A_3(f-1)}$ and so $N(r, a; f \mid \leqslant 1) \leqslant N(r, \frac{b(a-1)}{a}; \frac{A_2}{A_3}) = S(r)$, which contradicts Lemma 2.4(ii). Hence $A_3 + A_4 \not\equiv 0$.

Let $A_2 \equiv A_3 + A_4$. If $\frac{A_3 + A_4}{aA_3 + A_4} \equiv \frac{b}{a}$, then $\frac{A_4}{A_3} \equiv \frac{a(b-1)}{a-b}$ and so $g = \frac{(1+\frac{A_4}{A_3})f}{f + \frac{A_4}{A_3}} = \frac{b(a-1)f}{(a-b)f + a(b-1)}$, which is impossible because g is not a bilinear transformation of f. So $\frac{A_3 + A_4}{aA_3 + A_4} \not\equiv \frac{b}{a}$ and hence $N(r, a; f \mid \leqslant 1) \leqslant N(r, \frac{b}{a}; \frac{A_3 + A_4}{aA_3 + A_4}) = S(r)$, which contradicts Lemma 2.4(ii). Therefore $A_2 \not\equiv A_3 + A_4$.

Since $g = \frac{A_2 f}{A_3 f + A_4}$ and f, g share $(1,m)$, (∞,k), we get $N(r,\infty; f \mid \leqslant 1) \leqslant N(r,\infty; A_2) + N(r,0; A_3) + N(r,\infty; A_4) = S(r)$ and $N(r,1; f \mid \leqslant 1) \leqslant N(r,1; \frac{A_2}{A_3 + A_4}) = S(r)$. So by (i) of Lemma 2.3 we get $2T(r,f) = N(r,0; f \mid \leqslant 1) + N_0(r) + S(r)$ and so $N_0(r) \geqslant T(r,f) + S(r)$. This shows by (iv) of Lemma 2.3 and Lemma 2.7 that f is a bilinear transformation of g, which is a contradiction. Therefore $A_1 \not\equiv 0$.

Let $A_4 \equiv 0$. Then $g = \frac{A_1 f + A_2}{A_3}$ and so $A_3 \not\equiv 0$. If $A_2 \equiv 0$, then $\beta = \frac{A_1}{A_3}$, which contradicts (3.20). So $A_2 \not\equiv 0$. If $A_1 + A_2 \equiv A_3$, then we get $\alpha = \frac{f-1}{g-1} = \frac{A_3}{A_1}$, which contradicts (3.21). So $A_1 + A_2 \not\equiv A_3$.

Since $g \equiv \frac{A_1}{A_3}f + \frac{A_2}{A_3}$ and f, g share $(0,1)$, we see that a zero of f which is not a pole of $\frac{A_1}{A_3}$ must be a zero of $\frac{A_2}{A_3}$. Hence $N(r,0; f \mid \leqslant 1) \leqslant N(r,\infty; \frac{A_1}{A_3}) + N(r,0; \frac{A_2}{A_3}) = S(r)$.

Since f, g share $(1,m)$, we see that a simple 1-point of f is a 1-point of $\frac{A_1+A_2}{A_3}$. Hence

$$N(r,1;f\mid\leqslant 1)\leqslant N\left(r,1;\frac{A_1+A_2}{A_3}\right)=S(r).$$

Since by Lemma 2.10 $T(r,g)=T(r,f)+S(r)$, we get by (i) of Lemma 2.3 $2T(r,f)=N(r,\infty;f\mid\leqslant 1)+N_0(r)+S(r)$ and so $N_0(r)\geqslant T(r,f)+S(r)$. This shows by (iv) of Lemma 2.3 and Lemma 2.7 that f becomes a bilinear transformation of g, which is a contradiction. Therefore $A_4\not\equiv 0$.

Now from (3.23) we get by Lemma 2.10

$$T(r,g)=2T(r,f)+S(r). \tag{3.24}$$

Let $A_3+A_4\equiv 0$. Then $g\equiv f\frac{A_1f+A_2}{A_3(f-1)}$. Clearly $A_1+A_2\not\equiv 0$. For, otherwise $g\equiv\frac{A_1}{A_3}f$ and so by Lemma 2.10 we get $T(r,g)=T(r,f)+S(r)$, which contradicts (3.24). Since f, g share $(1,m)$, we see that $N(r,1;f\mid\leqslant 1)\leqslant N(r,\infty;A_3)+N(r,0;A_1+A_2)=S(r)$. Hence by (i) of Lemma 2.3 and (3.24) we get $3T(r,f)\leqslant N(r,0;f\mid\leqslant 1)+N(r,\infty;f\mid\leqslant 1)+N_0(r)+S(r)$ and so $N_0(r)\geqslant T(r,f)+S(r)$. Therefore by (iv) of Lemma 2.3 and Lemma 2.7 f becomes a bilinear transformation of g, which is a contradiction. So $A_3+A_4\not\equiv 0$.

Let $A_1+A_2\not\equiv A_3+A_4$. Since f, g share $(1,m)$ we get from (3.23) $N(r,1;f\mid\leqslant 1)\leqslant N(r,1;\frac{A_1+A_2}{A_3+A_4})=S(r)$ and so by (3.24) and (i) of Lemma 2.3 we get $N_0(r)\geqslant T(r,f)+S(r)$, which is impossible by (iv) of Lemma 2.3 and Lemma 2.7. So $A_1+A_2\equiv A_3+A_4$. Hence from the definition of A_j $(j=1,2,3,4)$ we get $B_2\equiv 0$ and so $A\equiv 0$. Since $\alpha\not\equiv 0$, we obtain from (3.18)

$$a\beta K+L\equiv 0.$$

So by (3.20) and the fact that $T(r,K)=S(r)$, $T(r,L)=S(r)$ we get from above $K\equiv 0$ and $L\equiv 0$. Therefore $\alpha''\equiv B\alpha'$ and $(\alpha\beta)''\equiv B(\alpha\beta)'$, where $B\not\equiv 0$. So $\frac{(\alpha\beta)''}{(\alpha\beta)'}\equiv\frac{\alpha''}{\alpha'}$ and on integration we obtain (3.22), which ultimately implies $T(r,\beta)=S(r)$, a contradiction to (3.20).

If $a_1=1$, $a_2=0$ and $a_3=\infty$, then we put $f_1=1-f$, $g_1=1-g$, $a_1=1-a$, $b_1=1-b$ and proceed as above. Also if $a_1=\infty$, $a_2=1$, $a_3=0$, then we put $f_2=\frac{1}{f}$, $g_2=\frac{1}{g}$, $a_2=\frac{1}{a}$, $b_2=\frac{1}{b}$ and proceed as above. Since m and k are interchangeable, we need not consider other permutations of a_1,a_2,a_3. Finally in these cases to get the specific forms as mentioned in the statement we need to replace the entire function γ by $\gamma+c$ for suitable constants c. This proves the theorem. $\square$

Proof of Theorem 1.2. We first note the following:

If f and g assume (x) of Theorem 1.1, then $f-a=(e^\gamma+\frac{1}{2})^2$ and $g-b=\frac{2(e^\gamma+\frac{1}{2})(1-e^\gamma)}{e^{2\gamma}}$.

If f and g assume (xi) of Theorem 1.1, then $f-a=\frac{(e^\gamma+2)^2}{e^\gamma+1}$ and $g-b=\frac{(e^\gamma+2)(1-e^\gamma)}{2e^\gamma(1+e^\gamma)}$.

If f and g assume (xii) of Theorem 1.1, then $f-a=-\frac{(2e^\gamma+1)^2}{3(1+e^\gamma+e^{2\gamma})^2}$ and $g-b=\frac{(2e^\gamma+1)(e^\gamma-1)}{3(1+e^\gamma+e^{2\gamma})}$.

If f and g assume (xiii) of Theorem 1.1, then $f-a=\frac{(e^\gamma+2)^2}{3(1+e^\gamma+e^{2\gamma})}$ and $g-b=\frac{(e^\gamma+2)(e^\gamma-1)}{3(1+e^\gamma+e^{2\gamma})}$.

If f and g assume (xiv) of Theorem 1.1, then $f-a=-\frac{(1+2e^\gamma)^2}{4}$ and $g-b=\frac{(2e^\gamma+1)(e^\gamma-1)}{e^{2\gamma}}$.

If f and g assume (xv) of Theorem 1.1, then $f-a=-\frac{(e^\gamma+2)^2}{1+e^\gamma}$ and $g-b=\frac{(e^\gamma+2)(e^\gamma-1)}{2e^\gamma(1+e^\gamma)}$.

If f and g assume (xvi) of Theorem 1.1, then $f-a=\frac{(e^\gamma-\frac{a\lambda}{2})^2}{\lambda e^\gamma-1}$ and $g-b=\frac{\lambda(e^\gamma-\frac{a\lambda}{2})\{e^\gamma-(\frac{2}{\lambda}-\frac{a\lambda}{2})\}}{e^\gamma-\lambda e^{2\gamma}}$.

If f and g assume (xvii) of Theorem 1.1, then $f-a=\frac{\{e^\gamma-2(1-a)\}^2}{4(a-1)(\lambda e^{2\gamma}-1)}$ and $g-b=\lambda\frac{\{e^\gamma-2(1-a)\}(e^\gamma-2a)}{2(\lambda e^{2\gamma}-1)}$.

If f and g assume (xviii) of Theorem 1.1, then $f-a=\frac{(e^\gamma+\frac{a-1}{2})^2}{\lambda-e^\gamma}$ and $g-b=\frac{(e^\gamma+\frac{a-1}{2})\{e^\gamma-(\frac{a-1}{2}+2\lambda)\}}{e^{2\gamma}-\lambda e^\gamma}$.

In view of Lemma 2.4(iii) we see from above that $\overline{E}_{1)}(b;g)\neq\phi$ and so the condition $\overline{E}_{1)}(b;g)\subset\overline{E}(a;f)$ implies that f and g do not assume the forms (x)–(xviii) of Theorem 1.1.

If f and g assume (iv) of Theorem 1.1, then $\overline{E}_{1)}(b;g)=\phi$ and so $ab=1$. This is the possibility (iv) of Theorem 1.2.

If f and g assume (vi) of Theorem 1.1, then $\overline{E}_{1)}(b; g) = \phi$ and so $ab = a + b$. This is the possibility (vi) of Theorem 1.2.

If f and g assume (viii) of Theorem 1.1, then $\overline{E}_{1)}(b; g) = \phi$ and so $a + b = 1$. This is the possibility (viii) of Theorem 1.2.

Other possibilities of Theorem 1.2 are those of Theorem 1.1. This proves the theorem. $\square$

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